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EXERCISE SHEET NO. 3 FOR COMPUTABILITY IN MATHEMATICS

Exercise 1. (Rice-Shapiro Theorem - 4 Points)

a) Let $A \subseteq B$ be two c.e. subsets of $\mathbb{N}$. Via reduction to the halting problem, show that no algorithm can, given the code of a r.e. set which is either A or B, stop iff this set is A.

b) Give a new proof of Rice's Theorem.

c) Show that if A_n is a sequence of uniformly c.e. sets, with $A_n \subseteq A_{n+1}$ for each n, and if $B = \bigcup A_n$, then no algorithm can, given the code of a c.e. set in $\{B\} \cup \{A_n, n \in \mathbb{N}\}$, stop iff this set is B.

For $A \subseteq \mathbb{N}$, denote by $I(A)$ the set:

$$I(A) = \{B \subseteq \mathbb{N}, B \text{ is r.e. and } A \subseteq B\}.$$

d) Prove the Rice-Shapiro Theorem:
Theorem.
A semi-decidable property of recursively enumerable sets is a union

$$\bigcup_{A \in M} I(A),$$

where M is a recursively enumerable sequence of finite sets.

Hint: Given a semi-decidable property P of c.e. sets, to define the corresponding M, just say that it is the set of finite sets that belong to P.

Exercise 2. (Undecidable by Rice - 2 Points)

Which of the following languages are undecidable due to Rice's theorem? If Rice's theorem is applicable, give the class of partial computable functions you apply the theorem to and a proof that it is non-trivial.

a) $\{e \in \mathbb{N} \mid \forall x : \varphi_e(x) \uparrow\}$

b) $\{e \in \mathbb{N} \mid (\forall x : \varphi_e(x) = x + 1) \vee (\forall x : x > 0 \Rightarrow \varphi_e(x) = x - 1)\}$

c) $\{e \in \mathbb{N} \mid \varphi_e(0) \text{ terminates after an even number of steps}\}$

d) $\{e \in \mathbb{N} \mid \forall x : \varphi_e(x) \downarrow \Rightarrow \varphi_e(x) = x + 1\}$

The solutions to this exercise sheet should be submitted until May 22nd at 10am in the letterbox no. 41. Please note full name and matriculation number on your submission.

Exercise 3. (The Listing Theorem is effective - 4 Points)

Prove that the Listing theorem is effective, i.e. there is a computable function $L : \mathbb{N} \to \mathbb{N}$ which for every non-empty c.e. set $A = \operatorname{dom} \varphi_e$ produces the total function with image A, meaning

$$\operatorname{im} \varphi_{L(e)} = \operatorname{dom} \varphi_e$$

Exercise 4. (Computably enumerable sets - 4 Points)

a) Prove that if A is c.e. and f is partial computable, then $f(A)$ and $f^{-1}(A)$ are c.e.

b) Let $A = \operatorname{im} \varphi_e$ for a total computable function φ_e and suppose there is a total computable function $\varphi_s : \mathbb{N}^k \to \mathbb{N}$ such that for all $a \in A$ there is $n \leq \varphi_s(a)$ such that $\varphi_e(n) = a$. Prove that A is in fact computable.

Exercise 5. (Computable subset of a c.e. set - 2 Points)

Show that every infinite computably enumerable set has an infinite computable subset.

Exercise 6. (Computably inseparable sets - 4 Points)

Consider the sets $A = \{e \mid \varphi_e(e) \downarrow= 1\}$ and $B = \{e \mid \varphi_e(e) \downarrow= 0\}$. Prove that they are **effectively inseparable**, i.e. there is no computable set C such that $A \subseteq C$ and $C \cap B = \emptyset$.

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